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Majority Rule at Low Temperatures on the Square and Triangular Lattices

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We consider the majority-rule renormalization group transformation applied to nearest neighbor Ising models. For the square lattice with 2 by 2 blocks we prove that if the temperature is sufficiently low, then the transformation is not defined. We use the methods of van Enter, Fernández, and Sokal, who proved the renormalized measure is not Gibbsian for 7 by 7 blocks if the temperature is too low. For the triangular lattice we prove that a zero-temperature majority-rule transformation may be defined. The resulting renormalized Hamiltonian is local with 14 different types of interactions.

KEY WORDS: Majority-rule renormalization-group transformation; non-Gibbsian measures.

1. INTRODUCTION

The majority-rule renormalization-group transformation is formally defined by

$$e^{-H'(\sigma')} = \sum_{\sigma} T(\sigma, \sigma')\, e^{-H(\sigma)} \tag{1}$$

Here σ are the spins in the original system and σ' are the block spins (or image spins). All spins are Ising spins, i.e., take on the values $-1, +1$. Here H is β times the original Hamiltonian, which throughout this paper will just be the ferromagnetic nearest neighbor Hamiltonian. $T(\sigma, \sigma')$ is the kernel for the majority rule transformation. If the lattice and block size are such that the number of spins in a block is odd, then this kernel only takes on the values 0 and 1. The kernel is 1 if in every block a majority of the

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spins agree with the block spin, and 0 otherwise. If the number of spins in a block is even, then there can be a "tie" in a block. In such systems the kernel is zero if there is a block with a clear majority and the majority disagrees with the block spin. If every block has either a clear majority which agrees with the block spin or a tie, then the kernel is equal to 2^{-n}, where n is the number of blocks in which a tie occurs. These factors of 1/2 are included so that $T(\sigma, \sigma')$ is a probability kernel, i.e., for every σ, $\sum_{\sigma'} T(\sigma, \sigma') = 1$. Equation (1) formally defines a new Hamiltonian H' for the block spins.

Equation (1) cannot be used to directly define $H'(\sigma')$ in the infinite-volume limit. One must use (1) for finite volumes and then try to take the infinite-volume limit. Another approach, emphasized in ref. 18, is to apply the majority rule transformation to infinite-volume Gibbs measures. The renormalized measure is always defined, but one must now deal with two problems. First, the original Hamiltonian may have more than one infinite-volume Gibbs measure. Second, the renormalized measure may not be the Gibbs measure of any reasonable Hamiltonian. Van Enter *et al.*[16, 18] used ideas of Griffiths and Pearce[5, 7] and Israel[9] to prove that at low temperatures the renormalized measure is non-Gibbsian for majority rule for a variety of block sizes, the smallest being 7 by 7. The idea behind the proof is to find a special block-spin configuration such that when the system of original spins is conditioned on this block-spin configuration, there is a phase transition. Loosely speaking, the strong correlations in the constrained original spin system then prevent the renormalized measure from being quasilocal.

There now exists a large collection of examples in which the renormalized measure is not Gibbsian. A review and extensive bibliography may be found in ref. 19. In many of these examples, including the case of 2 by 2 majority rule considered here, the trouble is caused by block-spin configurations (like the checkerboard one) that will never be seen at low temperature. Insisting that the renormalized measure be quasilocal uniformly in the block-spin configuration may be asking too much. By using a weaker definition of the renormalized Hamiltonian, it is often possible to prove that a renormalized Hamiltonian may be defined in cases where the renormalized measure is not uniformly quasilocal.[4, 11]

To show that the transformation is not defined for 2 by 2 blocks, we follow the method of ref. 18 closely. The special block-spin configuration that they use for 7 by 7 blocks is the "doubly alternating" configuration. This configuration consists of 2 by 2 groups of block spins of the same sign which alternate, i.e., for each such group of four block spins the four adjacent groups of four block spins have the opposite sign. With this block-spin constraint the original spins have two ground states. In one ground state most of the spins are $+1$, but there are 10 by 10 islands of -1 arranged so that the majority rule constraint is satisfied. The special block-spin configuration we use is the alternating or checkerboard configuration in which every pair of nearest neighbor block spins are not equal. For 2 by 2 blocks with this block-spin configuration we will show that the original spins have four ground states—the four "strip" states. Figure 1 shows one of them. Given the methods of ref. 18, the only nontrivial part of our proof is to show that these are indeed the ground states, a "Peierls condition" is satisfied, and a pure phase may be selected by a suitable choice of boundary conditions for the *block* spins. We do this by showing that the Hamiltonian with the majority rule constraint can be written as an "m-potential."[8] To verify that our rewritten form of the constrained Hamiltonian is indeed an m-potential, we enlist the help of a computer. There are close analogies between this paper and ref. 15. The constrained Ising model in a large field considered in that paper also has four striplike ground states. They must use a nonsquare volume to pick out a pure phase, a trick which we also employ.

Our final result is Theorem 4.5 of ref. 18 with "7×7" replaced by "2×2." For the convenience of the reader we restate the theorem. In the following, μT denotes the probability measure on the block spins, which is obtained from the Gibbs measure μ for the original spins and the transformation T in the usual way.[18]

Theorem 1. (For 7 by 7 blocks this is theorem 4.5 of ref. 18.) For all β sufficiently large, the following holds: Let μ be any Gibbs measure for

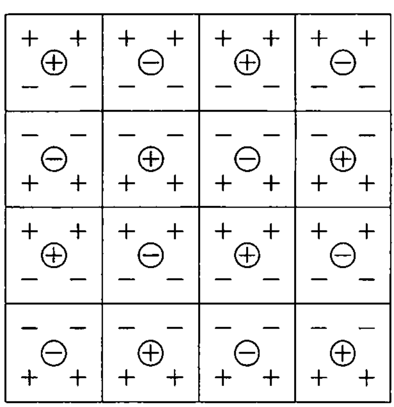

Fig. 1. Circled spins are block spins, uncircled spins are original spins. The block spins are in the checkerboard configuration, and the original spins are in one of the four ground states which we call "strip states."

the two-dimensional Ising model with nearest neighbor coupling β and zero magnetic field. Let T be the majority rule transformation on 2×2 square blocks. Then the measure μT is not consistent with any quasilocal specification. In particular, it is not the Gibbs measure of any uniformly convergent interaction.

Next we turn to our second result which concerns the zero temperature limit of the majority rule transformation. One approach would be to take a ground-state measure μ for H and then attempt to define H' by asking that μT be a ground-state measure of H'. However, many different Hamiltonians can give rise to the same ground-state measure, so this approach is not likely to yield a well-defined renormalized Hamiltonian. Our approach is to consider a finite volume and take the zero-temperature limit of (1) to obtain

$$H'(\sigma') = \min_{\sigma:\, T(\sigma,\sigma') \neq 0} H(\sigma) \tag{2}$$

[In (1) the inverse temperature β is hidden in H, so in taking this limit we must divide H' by β.] We then ask if this zero-temperature majority-rule transformation has an infinite volume limit, i.e., if $H'(\sigma')$ has an infinite-volume limit which belongs to some reasonable Banach space of Hamiltonians. If one looks at the argument which shows that the majority rule transformation is not defined at low temperature for 2 by 2 and 7 by 7 blocks, it is easy to adapt it to show that (2) does not have a nice infinite-volume limit in this case. For the triangular lattice the situation is quite different. We will prove that not only does (2) have an infinite-volume limit, but the renormalized Hamiltonian is a local function of the block spins. The precise result is as follows. A Hamiltonian is said to be local if it contains only a finite number of terms up to translations. In the following we work with finite volumes which are unions of blocks and which admit periodic boundary conditions. This last condition means that the finite volume and translations of it tesselate the lattice.

Theorem 2. For finite volumes Λ which admit periodic boundary conditions, define $H'_\Lambda(\sigma')$ by (2), where $H(\sigma)$ is the nearest neighbor ferromagnetic Hamiltonian for the original spins in Λ with periodic boundary conditions. There is a local translation-invariant Hamiltonian $H'(\sigma')$ on the block spins such that for sufficiently large volumes Λ, $H'_\Lambda(\sigma')$ equals the restriction of $H'(\sigma')$ to Λ with periodic boundary conditions. (The local renormalized Hamiltonian is given in Table 1.)

Of course, this theorem does not prove anything about majority rule on the triangular lattice for low but nonzero temperatures. However, it does show that the argument used to prove that the transformation is not defined for 7 by 7 and 2 by 2 blocks on the square lattice will not work for the triangular lattice. The theorem suggests the possibility that majority rule is actually defined for the triangular lattice at low temperatures, possibly for all temperatures.

2. SQUARE LATTICE WITH 2 BY 2 BLOCKS

Consider the checkerboard block spin configuration (ref. 18 calls this configuration the fully alternating configuration). We will show that with the constraint imposed by this block-spin configuration, the system of original spins has four periodic ground states. One of them is shown in Fig. 1. The other three are obtained by rotating this one by 90 deg and by applying a global spin flip to these two spin configurations. We will refer to these four states as the *strip states.*

A Hamiltonian H may be written in many ways as

$$H=\sum_{A} \Phi_{A} \tag{3}$$

where A is summed over finite subsets up to some fixed size and Φ_A is a function of the spins in A. Such a decomposition is said to be an m-potential if there is a configuration σ such that for every A

$$\Phi_A(\sigma)=\min \Phi_A \tag{4}$$

In other words, one can find a single configuration which simultaneously minimizes every term in the decomposition of the Hamiltonian.

Proposition 3. H may be written as an m-potential. Furthermore, the only configurations which simultaneously minimize every term in this representation of the Hamiltonian are the four strip states.

Proof. We will take the original Hamiltonian to be

$$H=\sum_{\langle ij\rangle} (1-\sigma_i\sigma_j)/2 \tag{5}$$

so that a pair of nearest neighbor spins that agree has energy 0 and a pair that disagrees has energy 1. Our representation of this Hamiltonian as an m-potential is rather complicated, so we will motivate it by showing why a natural simpler representation is not an m-potential. Divide the lattice into 4 by 4 squares so that each square contains four of the 2 by 2 blocks used by the majority rule.

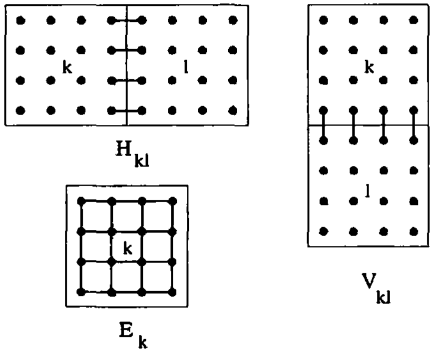

Fig. 2. The original Hamiltonian is the sum of the three types of terms E_k, H_{kl}, and V_{kl}. The block spins are not shown here.

Let E_k be the sum of the terms in (5) for which the bond $\langle ij \rangle$ is entirely in square k. For squares i and j which are "horizontally adjacent," i.e., they share a vertical edge, let H_{kl} be the sum of the terms such that one endpoint of the bond $\langle ij \rangle$ is in square k and the other in square l. For "vertically adjacent" squares k and 1, V_{kl} is defined similarly. Figure 2 shows E_k, H_{kl} and V_{kl}. With these definitions,

$$H = \sum_k E_k + \sum_{\langle kl \rangle : \mathrm{hor}} H_{kl} + \sum_{\langle kl \rangle : \mathrm{ver}} V_{kl} \tag{6}$$

The first sum is over squares k. The second is over horizontally adjacent squares k and l, and the third is over vertically adjacent squares k and l. Each pair of adjacent squares appears only once in the above. We make the convention that in H_{kl}, k is the left square and l is the right square. In V_{kl}, k is the upper square and l is the lower square. In a strip state, H_{kl} and V_{kl} are always zero, and $E_k = 8$. Figure 3 gives a configuration which gives a lower value for E_k, and thus this decomposition of the Hamiltonian is not an m-potential.

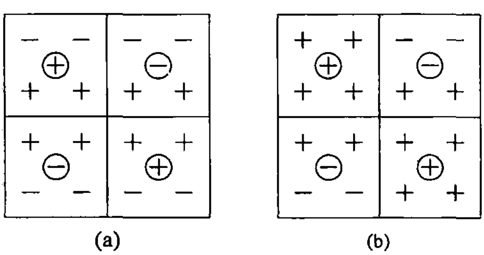

Fig. 3. Example showing that the strip states do not give the minimum of E_k. The configuration in (b) has lower E_k than the strip state shown in (a).

To modify the above decomposition of the Hamiltonian to give an m-potential, we introduce four functions L_k, R_k, U_k, D_k, each of which only depends on the spins in square k (L, R, U, D stand for left, right, up, down, respectively). Define

$$\hat{E}_k = E_k + L_k + R_k + D_k + U_k$$

$$\hat{H}_{kl} = H_{kl} - R_k - L_l$$

$$\hat{V}_{kl} = V_{kl} - D_k - U_l$$

Then we have

$$H = \sum_k \hat{E}_k + \sum_{\langle kl \rangle : \text{hor}} \hat{H}_{kl} + \sum_{\langle kl \rangle : \text{ver}} \hat{V}_{kl} \tag{7}$$

This equation holds for any choice of the functions L_k, R_k, U_k, D_k. Of course the hard part is finding a choice of these functions such that (2) is an m-potential. We will find a choice for which $\hat{E}_k \geqslant 8$, $\hat{H}_{kl} \geqslant 0$, $\hat{V}_{kl} \geqslant 0$, and these lower bounds are all attained by the strip configurations.

Some explanation of the left, right, up, down terminology is in order. Each of the functions L_k, R_k, U_k, D_k is a function of the 16 spins in the 4 by 4 square. However, they depend on these 16 spins only through the value of E_k and the four spins along one of the four edges of the square. Left, right, up, and down refer to which edge. Unfortunately, this causes some confusion when one considers $\hat{H}_{kl}$ and $\hat{V}_{kl}$. In $\hat{H}_{kl}$, k is the left square and l is the right square. Now, H_{kl} depends on the spins along the right edge of k and the left edge of l. So we subtract R_k and L_l in the definition of $\hat{H}_{kl}$.

We give the definition of R_k, along with a bit of motivation. The definitions of the other three functions are trivially obtained by rotation. We denote the four sites on the right edge of square k by 1, 2, 3, 4 (see Fig. 4). In the strip states the four spins $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ can be $(++++)$, $(----)$, $(+--+)$, or $(-++-)$. If $E_k < 8$ and the four spins agree with one of these four configurations, then $R_k = 0$. If $E_k < 8$ and the four spins do not agree with any of these four cases, then $R_k = 1$. If $E_k = 8$, then $R_k = 0$, regardless of the values of $\sigma_1, \sigma_2, \sigma_3, \sigma_4$. Note that in a strip configuration, $L_k = R_k = U_k = D_k = 0$.

The motivation for this part of the definition of R_k is to make $\hat{E}_k \geqslant 8$ for those configurations which have $E_k < 8$. As an example, consider Fig. 3(b). This configuration has $E_k = 6$. But $L_k = R_k = U_k = D_k = 1$, so $\hat{E}_k = 10$. If we completed the definition of R_k by defining it to be zero whenever $E_k > 8$, we would find that $\hat{H}_{kl}$ can be negative. An example is shown in Fig. 5. For

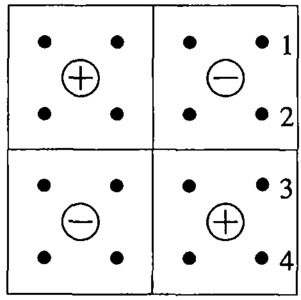

Fig. 4. The definition of R_k depends on the value of E_k and the four spins labeled 1, 2, 3, 4 in the figure.

the left square $E_k = 9$, and for the right square $E_l = 7$. Obviously $H_{kl} = 0$. But $L_l = 1$, so if R_k were defined to be 0, then $\hat{H}_{kl} = 0$ would be negative. To fix this problem we need to make R_k negative in some cases. Of course we must do this in such a way that $\hat{E}_k$ is still bounded below by 8.

We now complete the definition of R_k. If $E_k > 8$, we set $R_k = -1$ if three or more of the spins at 1, 2, 3, 4 are different from the block spin of the block in which they sit. Otherwise, $R_k = 0$. Note that R_k is still zero in all four strip states. We want to show that

$$\hat{E}_k \geqslant 8, \qquad \hat{H}_{kl} \geqslant 0, \qquad \hat{V}_{kl} \geqslant 0 \tag{8}$$

This is easily done on a computer. With the majority rule constraint, the number of allowable configurations on a 2 by 2 block is 11. So for a 4 by 4 block it is $11^4 = 14,641$. This is small enough that we can verify $\hat{E}_k \geqslant 8$ by simply computing every case. The number of cases to be checked for the second and third inequalities is $(14,641)^2 = 214,358,881$. Luckily, they do not all need to be checked. Since $H_{kl} \geqslant 0$, we have $\hat{H}_{kl} \geqslant 0$ if $R_k + L_l \leqslant 0$. This reduces the number of cases that must be explicitly computed to

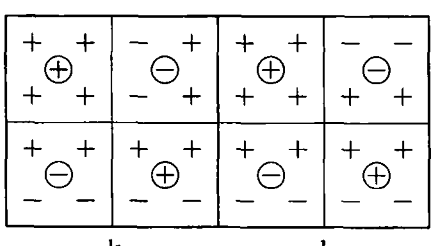

Fig. 5. An example which shows why we must make R_k negative for some spin configurations.

something manageable. The result of the computer program is that (8) is indeed true. Thus we have succeeded in writing the Hamiltonian as an m-potential.

Finally, we need to determine the ground states. The four strip states simultaneously minimize each of $\hat{E}_k$, $\hat{H}_{kl}$, and $\hat{V}_{kl}$. We will show that these are the only configurations that do this. We start by asking what configurations on a square have $\hat{E}_k = 8$. According to the computer, there are 233 such configurations, including the four strip states. We now consider a square k and the four squares adjacent to it. (We label them as follows: $i=$ up, $j=$ right, $l=$ down, $m=$ left.) We ask for the configurations on square k for which it is possible to find configurations on squares i, j, l, m such that

$$\begin{aligned} &\hat{E}_k = \hat{E}_i = \hat{E}_j = \hat{E}_l = \hat{E}_m = 8 \\ &\hat{H}_{kj} = \hat{H}_{mk} = \hat{V}_{ik} = \hat{V}_{kl} = 0 \end{aligned} \tag{9}$$

Again, we enlist the help of the computer. The answer is that there are only four such configurations on square k, the four strip states. Thus in a ground state every 4 by 4 square is one of the strip states. In the strip states $L_k = R_k = U_k = D_k = 0$, so (9) implies $H_{kj} = H_{mk} = V_{ik} = V_{kl} = 0$. This implies that in a ground state we have the same strip state in every 4 by 4 square. This completes the proof of Proposition 3. ∎

Let σ'_{cb} denote a checkerboard block-spin configuration. We have shown that when the Hamiltonian $H(\sigma)$ is restricted to σ with $T(\sigma, \sigma'_{cb}) \neq 0$, then $H(\sigma)$ is an m-potential with four ground states. It follows from a theorem of Holsztynski and Slawny[8] that this restricted H satisfies the sort of Peierls condition that one needs in order to carry out Pirogov–Sinai theory. Pirogov–Sinai theory implies that the original spin system conditioned on the checkerboard block-spin configuration will have four Gibbs states at low temperature that are small perturbations of the four ground states. The conditioned system we must consider is not simply given by restricting $H(\sigma)$ to the configurations with $T(\sigma, \sigma'_{cb}) \neq 0$. We must also add $-\ln T$ to the Hamiltonian. The Ising Hamiltonian comes with a factor of β, so $-\ln T$ is a small perturbation which can be handled by the Pirogov–Sinai theory. (Pirogov–Sinai theory is needed here rather than just a simple Peierls argument because the rotational symmetry of the lattice is being broken as well as the global spin-flip symmetry.) An introduction to Pirogov–Sinai theory in the context in which we need it may be found in Appendix B of ref. 18. The original reference is ref. 13. See also refs. 2, 10, 14, and 20.

Pirogov–Sinai theory establishes the phase transition that is responsible for the renormalized measure being non-Gibbsian. However, we are not finished. To apply the argument of ref. 18, we must show that one of the four pure phases may be selected by a suitable choice of boundary conditions. This would be easy if we were free to choose the boundary condition for the original spins. Unfortunately, we must show that this pure phase can be selected by a choice of boundary condition for the block spins. The special block-spin configuration we use to select a pure phase is shown in Fig. 6. Note that it is based on a finite volume in the shape of a parallelogram rather than a square. The two horizontal boundaries of the parallelogram favor one of the four strip states. The two boundaries at a 45 deg angle with respect to the lattice directions are neutral in the sense that each of the four strip states has the same boundary energy where it hits these boundaries. To prove that this block-spin configuration does indeed pick out one strip state, we show that the Hamiltonian can be written as an m-potential with a unique ground state. The strategy is the same as before, but we must modify the definition of the functions L_k, R_k, U_k and D_k in the vicinity of the boundary. The details are provided in the Appendix.

Finally, we need to show that by changing the boundary condition for the block spins, we can change the distribution of the block spins near the origin. This then implies that the renormalized measure cannot be quasilocal, i.e., is not consistent with any quasilocal specification. To do this we consider four block spins arranged in a square located near the origin. In the language of ref. 18, we "unfix" these four block spins. The four block spins we unfix are indicated in Fig. 6 by circles. Inside the four blocks the original spins will all be $+1$ with probability $1 - O(e^{-\beta})$. Let S denote the

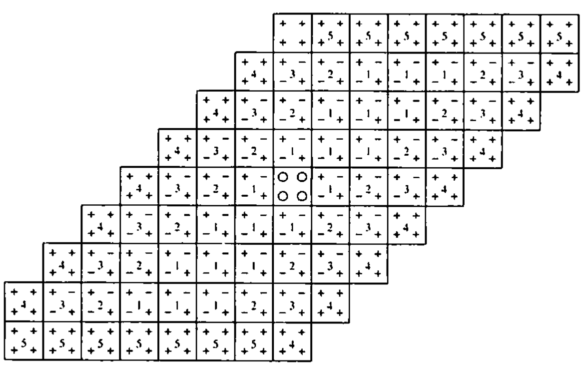

Fig. 6. The special block-spin configuration that picks out one of the pure phases for the original spin system conditioned on the checkerboard block-spin configuration. Note that only block spins are shown. We leave it to the reader to draw in the strip state that is picked out. The four block spins that are "unfixed" are indicated by circles.

set containing the sites of the four block spins that were unfixed. Let χ be the indicator function for the event that all four of these block spins are equal to $+1$. So $\chi = 1(\sigma'_i = +1, i \in S)$. Let E' denote expectation with respect to the renormalized measure μT on the block spins. $E'(\chi \mid S^c)$ denotes the conditional expectation of χ where we condition on the block spins not in S. This is a function of the block spins, so we write it as $E'(\chi \mid S^c)(\sigma')$. (Of course, it actually only depends on σ'_i with $i \notin S$.) Let U_+ be the set of block-spin configurations which agree with the block spin configuration shown in Fig. 6 and are arbitrary outside of the region shown in the figure. This is an open set in the product topology. We have shown that

$$E'(\chi \mid S^c)(\sigma') = 1 - O(e^{-\beta}), \qquad \forall \sigma' \in U_+ \tag{10}$$

Now suppose that we modify Fig. 6 as follows. We increase the height of the parallelogram by two block spins by moving each of the horizontal sides of the parallelogram by one block spin. The border of the parallelogram still consists of plus block spins, and the interior is the same checkerboard configuration as in Fig. 6. But now we will get the pure phase which is the global spin flip of the pure phase we had before. Hence the four block spins in S will be -1 with probability $1 - O(e^{-\beta})$. So

$$E'(\chi \mid S^c)(\sigma') = O(e^{-\beta}), \qquad \forall \sigma' \in U_- \tag{11}$$

where U_- is the set of block-spin configurations which agree with this modified version of Fig. 6 and are arbitrary outside of the parallelogram. The above estimates are uniform in the size of the parallelogram, so this proves that the conditional distribution of the four block spins we unfixed is essentially discontinuous and so cannot come from a quasilocal specification.

Our proof that the majority rule transformation is not defined at low temperature for 2 by 2 blocks is driven by the phase transition that takes place for the system of original spins with the constraint given by the checkerboard block-spin configuration. Our proof requires that the temperature be very low. However, one might expect that the transformation will not be defined for all temperatures below the critical temperature of this constrained system. Monte Carlo calculations of Ould-Lemrabott indicate that the critical β is approximately 1.0.[12] (For comparison, the critical β of the unconstrained Ising model is about 0.44.) Cirillo and Olivieri studied a slightly different majority rule transformation with 2 by 2 blocks.[3] When there is a tie in the block they take the block spin to equal the spin in the upper left corner of the block. They found that the critical β for the constraint of the checkerboard block spin configuration is approximately 1.6. Benfatto *et al.* did a Monte Carlo study of a renormalization group transformation in which the block spin is equal to the sum of the spins in the block.[1] For the block-spin configuration in which all the block spins are zero, they found that the constrained system's critical β was only about 10% higher than that of the original unconstrained model.

3. TRIANGULAR LATTICE AT $T=0$

In this section we prove Theorem 2. For the triangular lattice the blocks used by the majority rule are in fact triangles containing three sites. We will continue to refer to these triangles as blocks. Given a block-spin configuration, the ground state of the original spins subject to the majority rule constraint imposed by the block-spin configuration need not be unique. Luckily, this possible degeneracy (which can be rather large) does not concern us. To compute the minimum in (2), we need only find one ground state. We will give an algorithm for constructing one ground state. The algorithm will be local—the spin at a site is determined by the block spin for the block of that site and the block spins of the six blocks that surround that block.

Consider site i in Fig. 7. The four block-spin sites closest to site i are labeled I, J, K, L in the figure. We need to give these block spins names. We will refer to I as the "block spin of site i." Now suppose we stand at block spin I and face site i. Block spin J is the closest block spin in the forward direction, so we call it the "forward block spin of site i." Block spin K is the closest block spin to the left, so we refer to it as the "left block spin of site i." We call L the "right block spin of site i." We first show that it

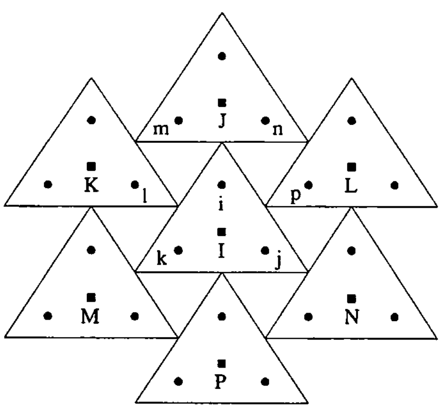

Fig. 7. Labeling of the original spins (circles) and block spins (squares) used in Proposition 4 and Table I.

is possible to find a ground state with the following property for every site. If at least one of the forward, left or right block spins agrees with the block spin of the site, then the spin at the site agrees with the block spin of the site. As an example, consider Fig. 7. If $\sigma'_I = +1$ and at least one of σ'_J, σ'_K, σ'_L equals $+1$, then σ_i is $+1$. (We are not claiming that every ground state has this property, only that at least one does.) In the following proposition we break this property into two properties for the sake of the proof.

Proposition 4. There is a ground state with the following properties.

(I) If i is a site with its block spin equal to its forward block spin, then σ_i equals the block spin of i.

(II) If i is a site with its block spin equal to either its left block spin or its right block spin, then σ_i equals the block spin of i.

Proof. We start by showing that there is a ground state with property *I*. It is enough to consider the case $\sigma'_I = \sigma'_J = +1$. Suppose $\sigma_i = -1$. By the majority rule constraint we must have $\sigma_j = \sigma_k = +1$. Since $\sigma'_J = +1$, the majority rule constraint implies that at least one of σ_m and σ_n is $+1$. Thus at least three of the nearest neighbors of σ_i are $+1$. So changing σ_i from -1 to $+1$ will not raise the energy. Applying this argument where needed, we obtain a ground state with property I.

Now we take a ground state with property I, and show that we can obtain a ground state with property II. It suffices to consider the case $\sigma'_I = \sigma'_K = +1$. By property I, this implies $\sigma_l = +1$. Suppose $\sigma_i = -1$. Then $\sigma_j = \sigma_k = +1$. Thus at least three of the nearest neighbors of σ_i are $+1$. So changing σ_i from $+1$ to -1 will not raise the energy. ∎

Given a block-spin configuration, the above two properties determine the ground state at a site unless the block spin of that site is different from all three of the forward, left and right block spins of the site. Consider Fig. 7 and define three conditions:

(Ci) $\sigma'_J = \sigma'_K = \sigma'_L = -\sigma'_I$.

(Cj) $\sigma'_N = \sigma'_L = \sigma'_P = -\sigma'_I$.

(Ck) $\sigma'_M = \sigma'_K = \sigma'_P = -\sigma'_I$.

If (Ci) does not hold, then the two properties in Proposition 4 determine σ_i. Likewise, they determine σ_j unless (Cj) holds, and determine σ_k unless (Ck) holds. Suppose that (Ci) holds and to be concrete consider the case $\sigma'_J = \sigma'_K = \sigma'_L = -1$ and $\sigma'_I = +1$. In this case, property I implies that $\sigma_m = \sigma_n = -1$. Property II implies $\sigma_l = \sigma_p = -1$. Thus at least four of the nearest neighbors of σ_i are -1. So if $\sigma_i = +1$, we can lower the energy by changing it to -1. However, if one of σ_j or σ_k is -1, then the majority rule constraint does not permit such a change. If neither of (Cj) or (Ck) hold, then $\sigma_j = \sigma_k = +1$, and so the ground state must have $\sigma_i = -1$. If two or more of (Ci), (Cj), or (Ck) hold, then for one of the sites for which the condition holds the corresponding spin must be -1. We showed above that when condition (Cx) holds, the four nearest neighbors of site x outside of the block containing x are all opposite to the block spin of x. So when two of these conditions hold, we will have the same energy no matter which site we choose to put the -1 at.

We now have an explicit algorithm for finding a ground state. Given the block spins $\sigma'_I, \sigma'_J, \sigma'_K, \sigma'_L, \sigma'_M, \sigma'_N, \sigma'_P$, the spins $\sigma_i, \sigma_j, \sigma_k$ are determined as follows.

1. If (Cx) does not hold, then set $\sigma_x = \sigma'_I$, where $x = i, j, k$.

2. If (Cx) holds and the other two of (Ci), (Cj), and (Ck) do not, then set $\sigma_x = -\sigma'_I$.

3. If two or more of (Ci), (Cj), and (Ck) hold, then set $\sigma_x = -\sigma'_I$ for one of the x for which (Cx) holds and set $\sigma_y = \sigma'_I$ for the other two sites. (This step is ambiguous, but we can remove the ambiguity by making some arbitrary rule for the choice of the site x.)

The above algorithm is local. To determine the value of an original spin at a site, we only need to know the values of the block spins in a neighborhood of that site. Thus the ground-state energy is a local function of the block spins. We compute it as follows. Consider the 12 blocks shown in Fig. 8. Given the values of the block spins for these 12 blocks, our algorithm determines the values of the original spins which are in the three inner blocks. The nine nearest neighbor bonds shown in the figure are

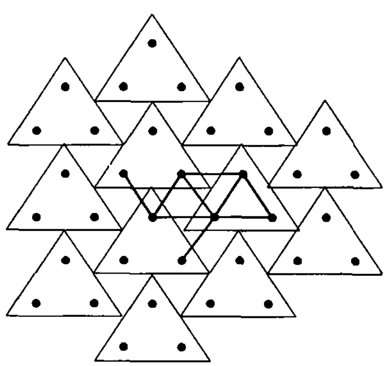

Fig. 8. Picture used in computing the local Hamiltonian H'.

chosen so that when they are translated by all translations commensurate with the block-spin lattice, we get every bond in the original lattice exactly once. Thus the ground-state energy $H'(\sigma')$ is obtained by computing the energy of these nine bonds and then summing over translations. Obviously the support of a term in H' must be a subset of the block spins shown in Fig. 8 or a translation of this set. In fact, we find that the only terms that actually appear in H' are those with support contained in some set of seven block spins arranged in a hexagon along with the block at the center of the hexagon. For example, the seven block spins shown in Fig. 7 are such a hexagonal set.

While the original triangular lattice is invariant under rotations by a multiple of 60 deg, the blocking partially breaks this symmetry and so H' need only be invariant under rotations by multiples of 120 deg. In addition to this rotational symmetry, the blocked lattice is also invariant under some reflections. Taking these symmetries into account, H' has 15 different terms. They are given in table I. Only one element from each symmetry class is given in the table.

The arguments we have given apply in any finite volume with periodic boundary conditions provided the volume is not so small that the sorts of regions we have been considering wrap back around on themselves. Thus we have proved Theorem 2.

Table I. Terms in the Local Hamiltonian H' [a]

Set of sites	Coefficient	Symmetry factor
I, J	26/16	3
L, K	2/16	3
J, P	−3/16	3
I, J, L, K	−4/16	3
J, L, P, K	−3/16	3
J, N, P, M	1/16	3
I, L, P, K	5/16	1
I, J, N, M	−1/16	1
I, J, L, P	3/16	6
J, L, N, P	−1/16	6
L, N, M, K	−1/16	3
I, J, P, M	1/16	6
J, L, N, P, M, K	1/16	1
I, L, N, P, M, K	1/16	3
I, J, L, N, M, K	−1/16	3

[a] The sites are labeled as in Fig. 7. Only one term from each symmetry class is shown.

APPENDIX

In this appendix we consider the system of original spins with the constraint imposed by the block-spin configuration shown in Fig. 6 from Section 2. We want to show that the Hamiltonian can be represented as an m-potential and has a unique ground state in the interior of the parallelogram. The strategy is the same as in Section 2. The only difference is in the definition of the functions L_k, R_k, D_k, and U_k. These functions now depend on where the square k is in relation to the parallelogram in Fig. 6. We divide the squares into five types, labeled 1–5, according to where the square is. See Fig. 6. The definitions of the functions are a bit involved. We will not attempt to motivate them; they were found mainly with trial and error and a little intuition. We should emphasize that they are by no means unique.

To define the functions L_k, R_k, U_k, D_k, we introduce a little notation. Recall that each of these functions is a function of a four by four square in the original lattice. For functions L_k, R_k, U_k, or D_k, the "four boundary spins" will refer to the four spins along the left, right, upper, or lower edge of the square, respectively. Each original spin belongs to a two by two majority-rule block. We will refer of the block spin of that two by two block as the block spin associated to the original spin. Now let n_∂ be the number of the four boundary spins that are not equal to their associated block spin. Let m_∂ be the number of the four boundary spins that are equal to -1. Finally we define a variable *strip* that takes on the values true and false. *strip* is true if the four boundary spins agree with the four boundary spins in some strip configuration, i.e., they must be one of $(++++)$, $(----)$, $(+--+)$, or $(-++-)$.

If square k is of type 1, then the functions L_k, R_k, U_k, D_k are all defined as follows. If $E_k < 8$ and *strip* is not true, then the value is $+1$. If $E_k > 8$ and $n_\partial > 2$, then the value is -1. Otherwise the value is 0.

Now consider a square k of type 2 near the left boundary. The functions R_k and D_k are defined as they were for type 1 squares. The functions L_k and U_k are defined as follows. If $E_k < 8$ and $n_\partial = 0$, then the value is $+1$. If $E_k > 8$ and $n_\partial > 2$ then the value is -1. Otherwise the value is 0. The definitions for a type 2 square near the right boundary are obtained in the obvious way by symmetry considerations.

For a square k of type 3 located near the left boundary the definitions are as follows. L_k and U_k equal m_∂ when $m_\partial \leqslant 3$ and equal 3 when $m_\partial = 4$. The definition of R_k and D_k is a bit more complicated when k is of type 3. If $E_k + L_k + U_k \leqslant 9$ and *strip* is not true, then their value is $+1$. If $E_k \geqslant 10$ and $n_\partial > 2$, then their value is -1. The definitions for a type 3 square near the right boundary are obtained in the obvious way by symmetry considerations.

If square k is of type 4 and near the left boundary, then R_k and D_k equal $-n_\partial$ when $n_\partial \leqslant 3$ and equal -3 when $n_\partial = 4$. The functions L_k, U_k are identically zero when k is of type 4. Again, symmetry determines the definitions when the square is near the right boundary.

If square k is of type 5 and near the top boundary, then D_k equals -1 when $E_k > 0$ and equals 0 otherwise. The functions L_k, R_k, U_k are identically zero. If square k is of type 5 and near the bottom boundary, then U_k equals -1 when $E_k > 0$ and equals 0 otherwise. The functions L_k, R_k, D_k are identically zero.

In the strip state selected by the block spin configuration in Fig. 6, we have $E_k = 8, 8, 10, 0, 0$ for k of type 1, 2, 3, 4, 5, respectively. We also have $H_{kl} = V_{kl} = 0$ and $L_k = R_k = U_k = D_k = 0$ in this strip state except for the following cases. If k is type 4 and l is type 3 and k is immediately left of l, then $H_{kl} = 2$ and $L_l = 2$. Thus in the strip state shown, $\hat{E}_k = 8, 8, 8, 0, 0$ for k of type 1, 2, 3, 4, 5, respectively, and $\hat{H}_{kl} = \hat{V}_{kl} = 0$. To prove that our decomposition of the Hamiltonian is an m-potential, we must show that these values are in fact the minimum of each of these functions. As in Section 2, this is easily done on the computer.

Finally we ask what are the ground states of the Hamiltonian. By the results of Section 2, in the interior of the parallelogram the configuration must be in one of the four strip states. Now consider the top horizontal edge of the parallelogram. For l of type 5 we have $E_l = 0$, so the original spins in the 4 by 4 squares of type 5 must all be $+1$. Now let k be a type 1 square just below a type 1 square l. Then $E_k = 8$ and so $U_k = 0$. We also have $D_l = 0$, so $\hat{V}_{kl} = V_{kl}$. But $\hat{V}_{kl} = 0$, so $V_{kl} = 0$. Since the spins in square l are all $+1$, it follows that a particular strip state is picked out for square k. The same argument applies to the bottom edge of the parallelogram. The height of the parallelogram is chosen so that the strip state picked out by the top edge is the same as the one picked out by the bottom edge. We have not shown that in a ground state the configuration must look like Fig. 6 along the edges of the parallelogram at 45 deg to the lattice directions. In fact, it need not. There are ways to modify Fig. 6 near these two edges that do not raise the energy. What we have shown is that any such modification cannot lower the energy, and away from these two edges the ground state must be the strip state picked out by the horizontal edges of the parallelogram.

To show that a Peierls condition is satisfied, we cannot appeal to the Holsztynski–Slawny theorem, since our system is not translation invariant. However, the same argument used in the proof of the theorem shows that the energy cost of a contour is at least proportional to the size of that part of the contour that does not run along the diagonal edges of the volume. Along the diagonal edges it is possible to have sections of contour that do

822/86/5-6-14

not cost any energy since the diagonal boundaries do not favor any one of the four ground states. However, for any contour the amount of the contour that can run along the diagonal edges is at most equal to the amount that runs along the top and bottom edges and inside the volume. Thus for any contour the energy associated with the contour will be proportional to the size of the contour.

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